STAT 2593

Lecture 007 - Counting Techniques

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Counting Techniques

Learning Objectives

1. Understand the product rule of counting.

2. Understand combinations, and when they are used.

3. Understand permutations, and when they are used.

	2		6	4		3	7	5	1	2	9
			1		3	9	6	4		5	
3	6	4	9	7	5						4
6	9		2	1	4	5	8	7			6
1		2		5	7	4	9	6		1	
4	7	5	8	6	9		3	2		4	7
2	1		5	8	6		4		2		1
7			4	9	1		5		6	7	3
5		8	7	3		6	1	9		8	5
3	2	4	1		8	9	6	5			4
	6	1	9		5		7	4		6	8
	5	7		4		8	2				9

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  - The challenge is in determining N_A, and sometimes in determining N.
- You likely already have intuition on some counting problems:
  - If there are K appetizers, ℓ mains, and M desserts, how many possible meals?
- The most basic rule of counting is the **product rule**.

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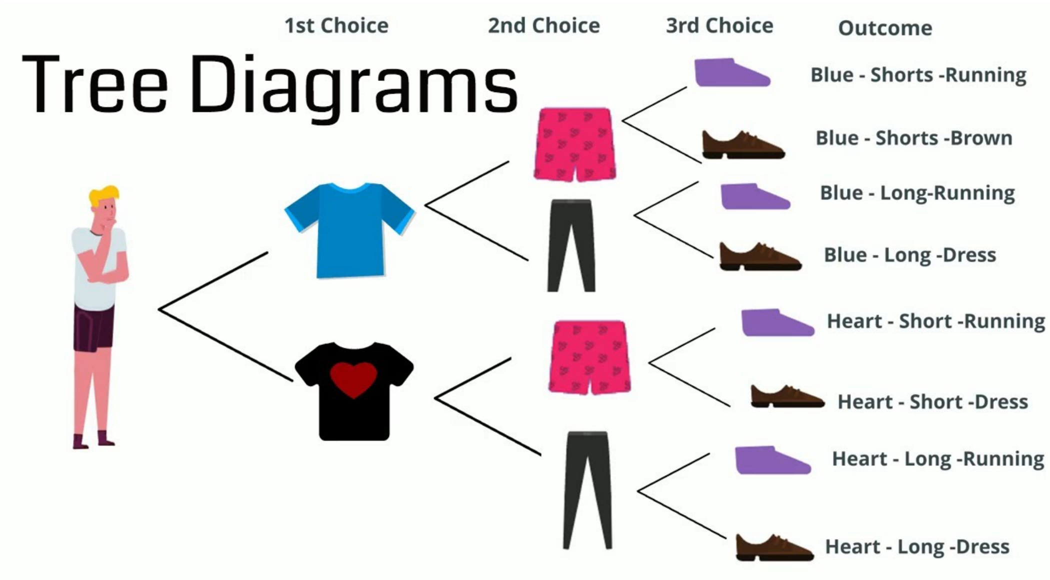
- Sometimes this is more useful to see via a tree diagram.



Source: https://www.youtube.com/watch?v=yxJi5vuO_wU

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- Thus, if there are 30 students in this class, then we can make $P_{10,30} = \frac{30!}{20!} = 109,027,350,432,000$ different lines of 10 students.

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- Thus, if there are 30 students in this class, then we can make $\binom{30}{10} = \frac{30!}{20!10!} = 30,045,015$ groups of 10 students.

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  - $\binom{30}{10} \times 10! = P_{10,30}$.
- Generally, to order k objects, we can use $k!$.

Summary

- To determine probabilities we must be able to count the occurrences of events.
- The simplest rule for counting is the product rule.
- We can use **permutations** to count **ordered** sets.
- We can use **combinations** to count **unordered** sets.
- Combinations and permutations are related to one another, through set ordering.